A STUDENT'S INTRODUCTION TO DIFFERENTIAL GEOMETRY AND GENERAL RELATIVITY

by

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Abstract

The study of light in physics started a serious revolution in the field. From the bending of light rays to the photoelectric effect, light continued to baffle scientists. The reason Einstein became so famous is because he was a revolutionary when thinking about light. It made complete sense to Einstein that massive objects in the universe could bend light as they reach earth. Einstein believed that the cause of this strange behavior of light was curvature of something called *spacetime*, and after a while, he convinced others as well. In order to fully understand this phenomenon, one must employ differential geometry and the theory of surfaces. We must thank Minkowski for his developments of differential geometry in the context of spacetime, as he was the first person to attempt studying spacetime as a geometric structure itself. Through the development of basic geometric procedures and physics background, we show how one can convince themselves that Einstein was right after all.

Acknowledgements

I would like to thank Dr. Stankewitz for the opportunity to learn from him during this project. Admittedly, differential geometry isn't the easiest subject to pick up and learn by yourself. His help throughout the semester was vital, as doing this on my own would have been extremely difficult. His vigilant reading of this thesis caught many silly mistakes, which I am very grateful for. I would also like to thank the physics department as a whole for the preparation to understand the physics of this project. It was very beneficial to only have to develop one whole subject at a time.

Author's Statement

The main reason I began this project was to (hopefully) make the entry into differential geometry and general relativity somewhat easier. I found myself struggling to find a resource that was understandable while learning. The main purpose of studying these subjects is to be able to fully describe what happens around us. The fact that light bends around a massive sun is extraordinary, but how do we go about explaining it? By studying geometry, we grant ourselves the ability to understand the structure of the universe at a fundamental level. This helps explain and model many of the events that occur out in the universe. Although I cannot answer every question and prove every theory in this thesis, a student familiar with calculus and modern physics should be prepared to read this and begin their journey to prove to themselves the results that Einstein has claimed for so long.

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1. INTRODUCTION

Becoming involved with spacetime or relativity is always a bit overwhelming at first. There are so many textbooks out there that attempt to ease a student into these confusing (yet exciting) fields, but oftentimes, they don't seem to be appropriate for undergraduate students. The hope of this thesis is to give an undergraduate student with basic experience in calculus and modern physics a solid introduction to these studies. By starting with the history, working through theory developed by many outstanding minds, and developing visual examples, the reader should feel appropriately acquainted with the concepts of spacetime and its related geometry. Hopefully the reader will become more interested and continue study into the future.

1.1 History

Einstein's theory of special relativity has been around since the miracle year of 1905. That year, among other things, Einstein released the paper "On the Electrodynamics of Moving Bodies", where special relativity was born. He had assistance, though, because Galileo had already proposed *relativity*: the concept that two observers moving uniformly with each other see physics the exact same way. Here, "uniformly" means without acceleration.

Einstein's most famous theory is born when he combines Galileo's theory of relativity with the assumption that the speed of light c is constant in all reference frames. Experiments by Albert Michelson and Edward Morley had already shown this was the case prior to special relativity, and theory by Hendrik Lorentz accounted for this discovery expressing that time actually *slows down* and lengths *contract* when moving near the speed of light [1]. The beauty of Einstein's work was this: he had shown that Lorentz' theories could actually be *derived* from accepting Galileo's relativity and the constancy of light's speed. In this sense, the idea of "spacetime" was born. No longer could the dimensions of space and the concept of time be separated, since both space and time are affected by speed.

What does this have to do with geometry? Well, a mathematician named Hermann Minkowski heard about this concept of spacetime from his former student Einstein. He realized that, since space and time cannot be separated, it would be more effective and realistic to treat this spacetime as a 4-dimensional geometry in and of itself. This step proved to be the basis of Einstein's home run work: general relativity. General relativity is inherently a different beast as compared to special relativity. As we will soon discover, special relativity does not adequately describe the universe when acceleration (gravity) is introduced. Let's begin with the Lorentz transformation.

2. THE LORENTZ TRANSFORMATION

In case the reader has not studied the Lorentz transformation recently, we will provide the details as a reminder because it is so central to the idea of relativity. If the reader is completely unfamiliar with the topic, there are a plethora of books that develop the theory of special relativity that could supplement when necessary [1] [2]. Remember, the Lorentz transformation is the description of what happens to space coordinates and time when some observer is moving *at a constant velocity* relative to another.

Imagine we have two frames: O that has spatial coordinates (x, y, z) and time coordinate t, while O' has spatial coordinates (x', y', z') and time coordinate t'. Here O' is moving with a constant velocity v with respect to O and so O' is considered inertial relative to O. To make things easier on ourselves, let's fix our coordinates so the motion is along the x/x' axis. See the figure below.

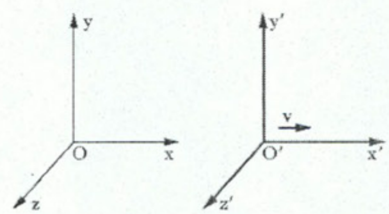

Fig. 2.1. O' moves at a constant speed v in order to be inertial [6].

The development of Lorentz transformations is meant for the ability to determine (x', y', z') and t' in O' while living in O or *vice versa.* This means an observer in frame O (seeing an event in terms of the variables x, y, z, t) can compute what an observer in the frame O' will see (in terms of x', y', z', t'). The transformation is given in the following equations:

1. $x' = \gamma(x - vt)$

2. $y' = y$

3. $z' = z$

4. $t' = \gamma\left(t - \frac{v}{c^2}x\right)$

where $\gamma := \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

These equations are usually developed with the x/x' axis being the direction of motion, so we only have a discrepancy between those spatial coordinates. That is, y and z are unaffected by the motion when there is no velocity in that direction. Note the role of γ here.

From typical kinematics, we would expect $x' = x - vt$ as some observer O' moved away at a constant velocity v. For $v << c$, as in our everyday experience, we see $\gamma \approx 1$ and so equation 1 above yields expected results (approximately). But, how do the transformation equations above behave as $v \to c$? We start to notice strange results: as $v \to c$ we have $\gamma \to \infty$. This is a major distinction between standard kinematics and the Lorentz transformation. Also note the effect of γ when v is relatively small, yet not negligible, compared to c. Suppose O' moves at a velocity of $v = 0.5c$, then $\gamma \approx 1.15$ and there is no extreme affect on our everyday kinematics (even at $v = 1.5 \times 10^8$ m/s)!

In order to produce the *inverse* Lorentz Transformation, i.e. using (x', y', z') and t' in O' to compute (x, y, z) and t in O, we have

5. $x = \gamma(x' + vt')$

6. $y = y'$

7. $z = z'$

8. $t = \gamma\left(t' + \frac{v}{c^2}x'\right)$

by switching the sign on the velocity v. These equations, namely 4 and 8, display the connectedness of space coordinates and time discussed briefly in the introduction.

2.1 Consequences

There are four famous consequences from these equations on physics. These consequences relate to length contraction, time dilation, simultaneity, and velocity addition. For our purposes, we will only be going over the first two because of their importance in what follows. For discussion on the latter two consequences, a special relativity textbook would be a good reference [1] [2].

2.1.1 Length Contraction

Let's say we are in the O' coordinate frame and we place a meter stick on the ground at the origin that extends to $L' \in O'$. Question: What is the length of the meter stick in O? Obviously the length *should be* a meter (equal to L' in this case). Let's examine the equations above and take our measurements at the same instant in time in the O frame, say $t = 0$, the moment when the origin of O and the origin of O' coincide.

Equation 1 dictates that one of the meter stick is at $x = 0$ and the other at $x = L'/\gamma$. So in O, any observer would see this "meter" stick to have a length of $L = L'/\gamma$. After verifying that $\gamma \geq 1$, this tells us that length is *contracted* by a factor of γ when an object is in motion. In O', we aren't witnessing the motion and hence do not see any contraction, but observers in O do.

2.1.2 Time Dilation

Now, instead of measuring the length of a meter stick, we will be using stop watches. Suppose we are in O' again and we are standing at the origin. Then we start our stop watch at $t' = 0$ and stop at $t' = T'$. According to equation 8 above, an observer in O would notice the stop watch start at $t = 0$ and end at $t = \gamma T'$. Again noting that $\gamma \geq 1$, we see that the time interval in O is longer. Another way of looking at this is the common saying "moving clocks run slow." In the next chapter we will uncover why these consequences, in particular, are important to the development to geometry and special relativity.

3. GEOMETRIC IMPLICATIONS

Let's begin by discussing the goals of geometry. In general, geometry attempts to understand and describe objects that do not change when transformed in some way. We typically call these objects *intrinsic* (to a given geometry) if they do not change. For example, imagine we are working in the Euclidean plane. If we have some vector $\overrightarrow{v} \in \mathbb{R}^2$ and we translate that vector we still have the same fundamental object in the standard Euclidean geometry.

We can start to make things more concrete by going about the basic tasks in plane geometry: measuring length and angles. This is done with the standard dot product. Let $\overrightarrow{v_1}, \overrightarrow{v_2} \in \mathbb{R}^2$ be regarded as column vectors, then our typical dot product is given by

$$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = \overrightarrow{v_1}^T \overrightarrow{v_2} = (x_1, y_1) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 x_2 + y_1 y_2, \tag{3.1}$$

which should already be quite familiar. Notice that we can measure the length of a vector in the plane (or $\mathbb{R}^n$ in general) using the dot product thusly

$$\|\overrightarrow{v}\| = \sqrt{\overrightarrow{v} \cdot \overrightarrow{v}} = \sqrt{x_1^2 + y_1^2}. \tag{3.2}$$

Of course, this is just the Pythagorean Theorem in $\mathbb{R}^2$. One of the fundamental aspects of Euclidean planar geometry is that lengths and angles are preserved under a rotation transformation. Recall the transformation matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.3}$$

By noting the effects on unit vectors like $(1, 0)$ and $(0, 1)$, one can verify that matrix (3.3) indeed gives a rotation about the unit circle. Now it is necessary to verify the claim that this rotation matrix preserves the length of vectors, i.e., $\|R_\theta \overrightarrow{v}\| = \|\overrightarrow{v}\|$ for all $\overrightarrow{v} \in \mathbb{R}^2$.

Let $R_\theta \vec{v} = (u, v)$ be the vector created by rotating some vector $\vec{v} = (x, y)$ in $\mathbb{R}^2$, i.e.,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we show $u^2 + y^2 = x^2 + y^2$, this would imply $\|R_\theta \vec{v}\|^2 = \|\vec{v}\|^2$, hence $\|R_\theta \vec{v}\| = \|\vec{v}\|$. Using the matrix equation above, we have

$$\begin{aligned} u^2 + v^2 &= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \\ &= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\ &= x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) \\ &= x^2 + y^2. \end{aligned}$$

As we suspected, it is true that rotating a vector in the plane does not change its length. This is the benefit of the dot product: it preserves important properties like length when undergoing standard manipulations such as rotations. But what about transformations that aren't as standard as rotations? In particular, what about the Lorentz transformation?

We discovered in the previous section that lengths are not actually conserved under this transformation, suggesting that the standard dot product is no longer a valid way of attempting to measure lengths. In order to verify this suspicion, we are going to introduce some four-vector notation that is commonly used in relativistic kinematics that will help in developing a "new" dot product.

3.1 Four-Vector Notation

After seeing the Lorentz transformation, we start to get a feel for spacetime as the coupling of space coordinates and time in the equations. Let's define some objects that will condense the Lorentz equations and get us moving in the right direction. Let the *position-time* four-vector x_μ and x'_μ, with $\mu = 0, 1, 2, 3$, be defined as

$$x_0 = ct, \qquad x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \tag{3.4}$$

$$x'_0 = ct', \qquad x'_1 = x', \qquad x'_2 = y', \qquad x'_3 = z'. \tag{3.5}$$

Note that we can now rewrite the Lorentz equations as

1. $x'_0 = \gamma(x_0 - \beta x_1)$
2. $x'_1 = \gamma(x_1 - \beta x_0)$
3. $x'_2 = x_2$
4. $x'_3 = x_3$

where $\beta = v/c$. It is helpful to remember that the zero-th component term x_0 refers to the time "coordinate", while the other components x_1, x_2, x_3 all refer to the standard spatial coordinates. We build this system intentionally, as it allows us to quickly establish a matrix that describes the transition from x_μ to x'_μ. Verify that equations (1) through (4) above can be written as

$$\begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = L \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$L = \gamma \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Of course, we usually try to make work easier for ourselves. By noting that L effectively does nothing to the $x_2 = y$ and $x_3 = z$ components, we can use the following simplified matrix for L when studying the x_0 and x_1 components by themselves:

$$L = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

3.2 A New Measuring Tool

Let's remind ourselves what our current goal is. We are trying to determine what kind of "dot product" can be used to keep measuring "lengths" even when Lorentz transformations modify lengths and time. When dealing with the rotation matrix R_θ, we simply established a matrix equation and noticed that dot product conservation came out in the wash. Let's try the same here. Let the $x_1 = x$ be the direction of motion with velocity $v = \beta c$, so we have the matrix equation

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

I won't be asking you to come up with some sort of conserved product. Instead, I will give you a hint: investigate the quantity ${x_0'}^2 - {x_1'}^2$ by noting

$$\begin{aligned}
{x_0'}^2 - {x_1'}^2 &= [\gamma(x_0 - \beta x_1)]^2 - [\gamma(-\beta x_0 + x_1)]^2 \\
&= \gamma^2(x_0 - \beta x_1)^2 - [\gamma^2(-\beta x_0 + x_1)^2] \\
&= \gamma^2[(x_0^2 - 2\beta x_0 x_1 + \beta^2 x_1^2) - (\beta^2 x_0^2 - 2\beta x_0 x_1 + x_1^2)] \\
&= \gamma^2[x_0^2(1 - \beta^2) + x_0^2(\beta^2 - 1)] \\
&= \gamma^2(1 - \beta^2)[x_0^2 - x_1^2] \\
&= x_0^2 - x_1^2
\end{aligned}$$

since $\gamma^2(1 - \beta^2) = 1$. Now that's pretty neat. Our standard dot product would have looked like ${x_0'}^2 + {x_1'}^2 = x_0^2 + x_1^2$, but due to length contraction in Section 2.1.1, this no longer holds. This is telling us quite a bit: when undergoing the Lorentz transformation L and when using spacetime coordinates x_μ, we should really be utilizing ${x_0'}^2 - {x_1'}^2$ as our tool for measuring "distance". This was exactly the procedure adopted by Minkowski when developing the theory of spacetime from a geometric standpoint, which is why he has become very famous in the field. One can easily verify that the quantity $x_0^2 - x_1^2 - x_2^2 - x_3^2$ is also conserved by applying the same method above to the full 4×4 matrix L.

3.3 The Minkowski Norm and the Light Cone

On the topic of our new measuring tool $x_0^2 - x_1^2$, note that this quantity can be positive, negative, or zero. This is completely unlike the standard dot product, which is always non-negative. This led people like Minkowski to study, for an $E = (x_0, x_1, x_2, x_3)$, the following quantity

$$Q(E) = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{3.6}$$

Note that (3.6) is working entirely in the full 4-dimensional spacetime. One could easily study $Q(E) = x_0^2 - x_1^2 - x_2^2$ in order to stay within $\mathbb{R}^3$, for instance. The set of all events is usually partitioned by the following categories:

a. If $x_0^2 > x_1^2 + x_2^2 + x_3^2$, then $Q(E) > 0$ and E is considered *timelike.*

b. If $x_0^2 < x_1^2 + x_2^2 + x_3^2$, then $Q(E) < 0$ and E is considered *spacelike.*

c. If $x_0^2 = x_1^2 + x_2^2 + x_3^2$, then $Q(E) = 0$ and E is considered *lightlike.*

We will be using the concept of regions in spacetime, in particular the *spacelike* portion, during Chapter 6 when dealing with DeSitter Spacetime. In the meantime, we can brainstorming what kind of shapes and sets are formed by equation (3.6) when $Q(E) > 0$, $Q(E) < 0$, and $Q(E) = 0$. Instead of working in $\mathbb{R}^4$, which can be difficult to visualize, we will truncate (3.6) appropriately for something in $\mathbb{R}^3$ like we mentioned before. We can depict these sections by generating what is called the light cone, shown below with only two spatial coordinates y and z and time coordinate t.

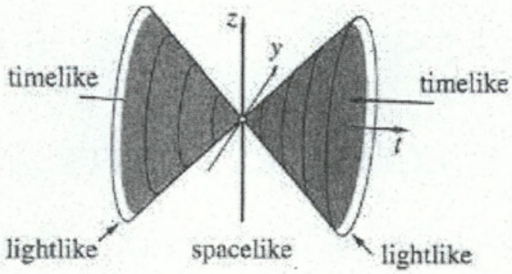

Fig. 3.1. The light cone in $\mathbb{R}^3$.

Instead of an event E in this spaceime being a standard *Euclidean* distance from the origin, we give the event E a **Minkowski norm** as defined by

$$\|E\| = \begin{cases} \sqrt{Q(E)} & \text{if } E \text{ is } \textit{timelike}\text{: } Q(E) > 0, \\ \sqrt{-Q(E)} & \text{if } E \text{ is } \textit{spacelike}\text{: } Q(E) < 0, \\ 0 & \text{if } E \text{ is } \textit{lightlike}\text{: } Q(E) = 0. \end{cases}$$

This is reasonable, based on the results from Section 3.2. There we learned that the standard dot product, and hence the standard Euclidean distance, is not conserved while undergoing the Lorentz transformation. Naturally, this is why we operate with the Minkowski norm while working in spacetime. This distinction is very important when we develop DeSitter Spacetime.

4. SPECIAL RELATIVITY AND GRAVITY

The development of special relativity has been great, but remember the one downfall to the entire scheme: *inertial* frames are the only valid frames in the theory. As a reminder, inertial frames are reference frames (or coordinate frames, if you like) that are not accelerated in any way with respect to O. In the Lorentz equations, we required that the O' frame was moving *at a constant velocity.*

What's the big deal anyways? For one, the most basic equation in all of physics $\vec{F} = m\vec{a}$ is stowed away in this context. We can no longer consider forces that cause acceleration in special relativity. Okay, let's say we are studying a system which has no net acceleration. What about gravity? This is the one shortcoming of special relativity. This issue persisted until the creation of general relativity to deal with it, since a gravitational field necessarily implies acceleration. Even if a frame is moving at a constant velocity (which is already somewhat boring), we will see that we cannot escape the grasp of a gravitational field tampering with special relativity. Then, we will discuss the possibilities of modifying special relativity to allow for gravity in the theory. What's the worst that can happen?

4.1 Equivalence Principle

Let's start with a thought experiment. Everyone has dropped something at a point in their life. What caused this? "Gravity, obviously", most would say. After all, everyone knows the classic story of an apple falling on Newton's head due to gravity. But how about this: let's now imagine we are flying in a region of space so far from celestial bodies that external gravitational fields are nearly zero.

If we happen to be accelerating in a spaceship at about $a = 9.8$ m/s^2, what would it look like if you dropped an apple? The apple isn't aware that it is in a spaceship, it simply experiences a force roughly equal to gravity on Earth's surface. Now, how would you go about proving you were in fact in a spaceship instead of standing on the surface of the Earth? Without utilizing external means such as looking out of a window leading to outer space, there seems to be no way of showing that the acceleration you are sensing is coming from propulsion rather than a gravitational field.

This is the core concept of Einstein's Equivalence Principle: In small enough regions of space, it is impossible to come up with a local experiment that detects and proves the existence of a gravitational field since any acceleration can feel just like gravity. One of the most praised predictions of Einstein's Equivalence Principle is the gravitational redshift of photons in space.

4.2 Gravitational Redshift

Let's return to our spaceships for one moment. This time let's say there are two spaceships, both accelerating at the same rate along the same direction. At time t_0, the trailing spaceship emits a photon of wavelength λ_0. Since the spaceships are at a constant distance apart, the photon will arrive at the leading spaceship in a time of $\Delta t = x/c$, where x is the distance between ships. During the short travel time of the photon, the spaceships will increase in velocity by $\Delta v = a\Delta t$ due to their acceleration a.

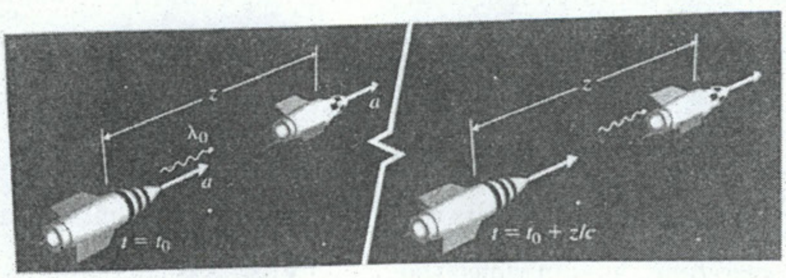

Fig. 4.1. The photon of original wavelength λ_0 travels between spaceships as their velocities increase by Δv [5].

This will cause a standard Doppler effect on the photon's wavelength, expressed by

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{ax}{c^2}. \tag{4.1}$$

We should pay close attention to the acceleration a in equation (4.1). As we just discussed in the previous section regarding the Equivalence Principle, the acceleration experienced by these spaceships is exactly identical to the effect from a gravitational field. That is, if a photon were emitted in a gravitational field from one stationary point to another, we should be able to detect a change in wavelength given by (4.1) where $a = g = 9.8\ \mathrm{m/s^2}$.

Experiments by Pound and Rebka at Harvard tested this suspicion in 1959 [2]. The easiest way to test this hypothesis is to emit a photon on the ground upwards toward a tower with some sort of absorption device. The individual on the ground would measure a time interval of a photon traveling one wavelength to be $\Delta t_0 = \lambda_0/c$. As for the individual at the top of the tower, they would measure a time interval of the same photon traveling one wavelength to be $\Delta t_1 = \lambda_1/c$. According to equation (4.1), if x is the height of the tower, we will have

$$\Delta\lambda = \frac{gx}{c^2}\lambda_0 > 0$$

and hence $\Delta t_1 > \Delta t_0$. Pound and Rebka's experiment confirmed these results as well.

The important note is this: according to our standard geometry, there should be no reason that a photon changes wavelength when it travels a constant distance. Physics has been aware of Doppler effects for some time, but this has always been the effect of some relative motion between frames. In the experiment of the tower, the *spatial* distance between the emission and absorption points are constant. How can we reconcile these results while claiming the constant speed of light? Geometry. The *spacetime* through which the photon travels is curved in the presence of gravity! We can no longer live in flat Euclidean geometry if we wish to understand the universe. We must utilize the tools and ideas of differential geometry and curvature to completely describe this new world.

5. DIFFERENTIAL GEOMETRY

I will remind the reader of the underlying goals of geometry: to understand *intrinsic* properties of structures, that is, to define surfaces as a transformation of regular planes with their own special measuring tools in order to determine curvature, lengths, etc. To see the full benefit of this, I believe it is helpful to see an example.

Consider the equilateral triangle below, which is meant to represent a quarter of the northern hemisphere. We draw this from the perspective of our friend Bob walking these lengths. Supposing Bob is unaware that the earth is round, he would likely map his path as such, i.e., as if it all happened in a flat plane.

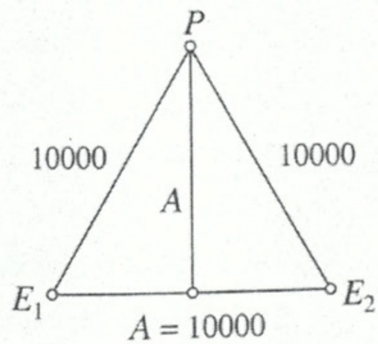

Fig. 5.1. Flat, triangular region that Bob would draw [2].

In order to find the length of A, Bob then employs simple *Euclidean* trigonometry to conclude that the length of the vertial segment $A = 10,000(\sqrt{3}/2) = 8,660$ km. However, when Bob walks this path he *measures* it to be $A = 10,000$ km, and is quite surprised that it is so far from his calculation. Watching all of this from her spaceship hovering above the earth, Alice is not surprised at all. She sees that Bob is walking on the *curved* surface of the earth and knows that standard Euclidean geometry in the plane does not apply. She would see Bob's path on the surface look something like the hemisphere below.

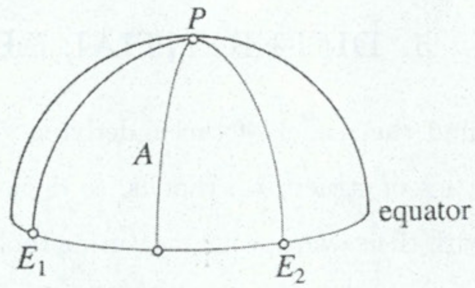

Fig. 5.2. Bob's true path, curving over the surface of earth [2].

Alice has an *extrinsic* view and can "see" the curvature. Bob, being stuck on the 2-dimensional surface of the earth, does not have this view. Bob can detect or measure what he can (and should) call curvature simply because his Euclidean geometric calculations are different from his measurements. Bob's perspective is called *intrinsic* because he can detect curvature without needing a higher dimensional perspective, like Alice's spaceship.

This is exactly what we accomplished with the gravitational redshift experiment – the change of wavelength in the photon between two stationary points suggested that spacetime was curved. This is an inherently *intrinsic* perspective, because we did not step outside of spacetime in order to watch the photon travel over some sort of curved structure.

5.1 Basic Geometric Procedures

Here we will include some important results from Calculus and differential geometry that will be necessary for us. If at any point the reader feels completely lost, there are great options for further detail and reference [3] [4].

5.1.1 Constructing a Surface

Generating a surface in $\mathbb{R}^3$ is all about distorting a standard plane in some way. In mathematical terms, that means that there is a map $\vec{x}$ that will take points from $\mathbb{R}^2$ and places them in $\mathbb{R}^3$. We can write this as $\vec{x} : D \to \mathbb{R}^3 : (u, v) \to (x(u, v), y(u, v), z(u, v))$, where D is a region in the uv-plane. Our mapping $\vec{x}$ will be defined by smooth functions of u and v as

$$\vec{x}(u, v) = (x(u, v), y(u, v), z(u, v)). \tag{5.1}$$

Our generated surface S is the image of $\vec{x}$, i.e., $S = \vec{x}(D)$. The function $\vec{x}$ is called a *parametrization* of the surface S. This procedure is shown graphically below.

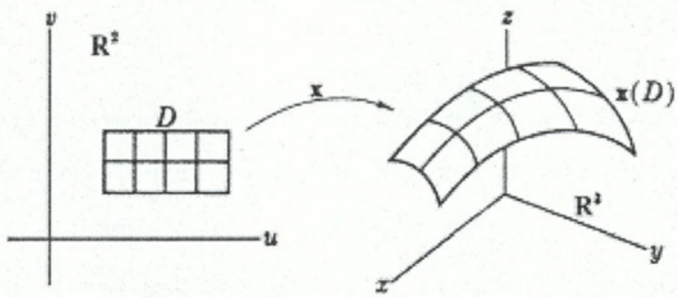

Fig. 5.3. The map $\vec{x}$ takes D and places it inside of $\mathbb{R}^3$ [4].

It is important to have our bearings straight while on the surface S. We are operating on a new structure, and we need to know how to move about on the surface. In the Euclidean plane, we have the standard basis vectors $\vec{i}$ and $\vec{j}$ that acted as our cardinal directions, but those no longer apply on S. Navigation on S is typically accomplished by using curves generated by the parameters u and v. Let's say we hold v constant as $v = v_0$, while u varies in the uv-plane. Our map from (5.1) is evidently $\vec{x}(u, v_0)$ – just a curve in $\mathbb{R}^3$ with u as the parameter. We can just as easily form v-parameter curves $\vec{x}(u_0, v)$ by holding u constant as $u = u_0$. The curves give us the ability to calculate vectors tangent to the surface, identical to velocity vectors from Calculus and Mechanics. These tangent vectors are given by

$$\vec{x}_u = \partial_u \vec{x}(u_0, v_0) \qquad \text{and} \qquad \vec{x}_v = \partial_v \vec{x}(u_0, v_0) \tag{5.2}$$

where (u_0, v_0) is a position in the uv-plane. Notice here that these tangent vectors can change depending on the values u_0 and v_0.

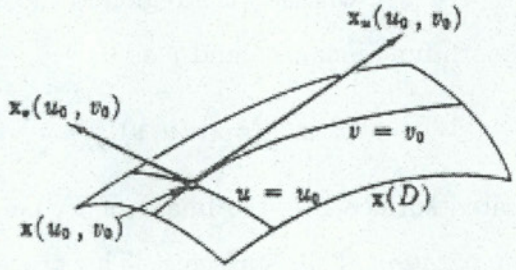

Fig. 5.4. Tangent vectors shown on the surface $S = \vec{x}(D)$

Note that it is not necessary for $\vec{x}_u$ and $\vec{x}_v$ to be orthogonal. However, we will focus our study on **regular surfaces**, which are defined as mappings such that $\vec{x}_u \times \vec{x}_v$ is never zero. Here the $\times$ signifies the standard cross product. One benefit of restricting ourselves to regular surfaces it that we will always have an available *unit normal vector* $\vec{n}(c_1, c_2)$ that can be defined as

$$\vec{n}(c_1, c_2) = \frac{(\vec{x}_u \times \vec{x}_v)(c_1, c_2)}{\|(\vec{x}_u \times \vec{x}_v)(c_1, c_2)\|}. \tag{5.3}$$

Because of the restriction to *regular surfaces*, the tangent vectors $\vec{x}_u$ and $\vec{x}_v$ will be linearly independent, thus always form a basis for a 2-dimensional vector space. We will call this vector space the tangent space $T_p(S)$, where p is a position on S. An example of a tangent space is given below.

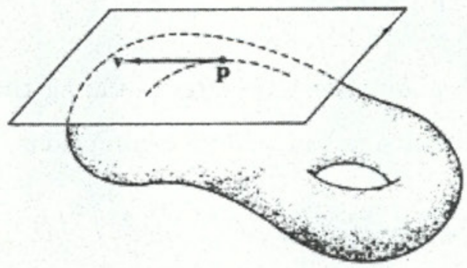

Fig. 5.5. The tangent space at a point $\vec{p}$ on the surface [4].

Now we can start to investigate vectors in the tangent space. Let $\vec{v}, \vec{w} \in T_p(S)$. This means $\vec{v}$ and $\vec{w}$ can both be expressed as linear combinations of $\vec{x}_u$ and $\vec{x}_v$ as

$$\vec{v} = v_1\vec{x}_u + v_2\vec{x}_v \qquad \text{and} \qquad \vec{w} = w_1\vec{x}_u + w_2\vec{x}_v.$$

It is so important to have an idea as to what this tangent space does for us. Think of this: when you stand on the Earth, does it feel round? No, a level ruler that is precise cannot sense the curvature of the Earth. This is why our friend Bob cannot detect curvature based only on what he "sees", but must rely on calculations instead. Does this mean the Earth must be flat? Again, no, and we will investigate this when we have the necessary tools. The tangent space acts as the space that inhabitants *on the surface* would be living in. For us on Earth, we experience a flat local geometry on a giant sphere. No matter the shape of a regular surface S in $\mathbb{R}^3$, there will be an associated tangent space $T_p(S)$ that describes the geometry for inhabitants on S. So, our goal is to be able to detect curvature *while living in* $T_p(S)$, which is just a copy of $\mathbb{R}^2$. To do this, we must develop the **Metric**.

5.1.2 Defining the Metric

Allow me to go about the procedure of taking the classic dot product of $\vec{v}$ and $\vec{w}$, while being careful to not make any assumptions:

$$\begin{aligned}
\vec{v}\cdot\vec{w} &= (v_1\vec{x}_u + v_2\vec{x}_v)\cdot(w_1\vec{x}_u + w_2\vec{x}_v)\\
&= v_1w_1(\vec{x}_u\cdot\vec{x}_u) + v_1w_2(\vec{x}_u\cdot\vec{x}_v) + v_2w_1(\vec{x}_v\cdot\vec{x}_u) + v_2w_2(\vec{x}_v\cdot\vec{x}_v)\\
&= (v_1, v_2)\begin{pmatrix}\vec{x}_u\cdot\vec{x}_u & \vec{x}_u\cdot\vec{x}_v\\ \vec{x}_v\cdot\vec{x}_u & \vec{x}_v\cdot\vec{x}_v\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}\\
&= \vec{v}^T G\vec{w}.
\end{aligned}$$

where G can be written as

$$G = \begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} = \begin{pmatrix}\vec{x}_u\cdot\vec{x}_u & \vec{x}_u\cdot\vec{x}_v\\ \vec{x}_v\cdot\vec{x}_u & \vec{x}_v\cdot\vec{x}_v\end{pmatrix} = \begin{pmatrix}\|\vec{x}_u\|^2 & \vec{x}_u\cdot\vec{x}_v\\ \vec{x}_v\cdot\vec{x}_u & \|\vec{x}_v\|^2\end{pmatrix}. \tag{5.4}$$

The name for G is the **metric** for $T_p(S)$.

I mentioned that I would do this operation without making assumptions, because we are used to only two terms in a Euclidean dot product. Even though we are operating in $\mathbb{R}^2$, we cannot make any assumptions about $\vec{x}_u\cdot\vec{x}_v$. Remember, these need not be orthogonal. To see the connection between G and our typical Euclidean geometry, let's see a quick example.

5.1.3 Example: The Euclidean Plane Metric

In order to form a surface, we actually need to generate a map $\vec{x}$ in the form of (5.1) that creates a copy of $\mathbb{R}^2$. Let's try this:

$$\vec{x}(u,v) = (u, v, 0). \tag{5.5}$$

Evidently, we will have tangent vectors given by

$$\vec{x}_u = (1,0,0) \qquad \text{and} \qquad \vec{x}_v = (0,1,0).$$

Notice that $\vec{x}_u = \vec{i}$ and $\vec{x}_v = \vec{j}$, our standard basis vectors we are so familiar with. Our Euclidean **metric** is then

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \vec{i} \cdot \vec{i} & \vec{i} \cdot \vec{j} \\ \vec{j} \cdot \vec{i} & \vec{j} \cdot \vec{j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, the Euclidean metric is just given by the identity matrix. This is what has caused many of our familiar results so easy, because $\vec{x}_u \cdot \vec{x}_v = 0$ when constructing the "surface" of the plane. We can use Mathematica to double check that our map in (5.5) actually generates the plain like I have claimed.

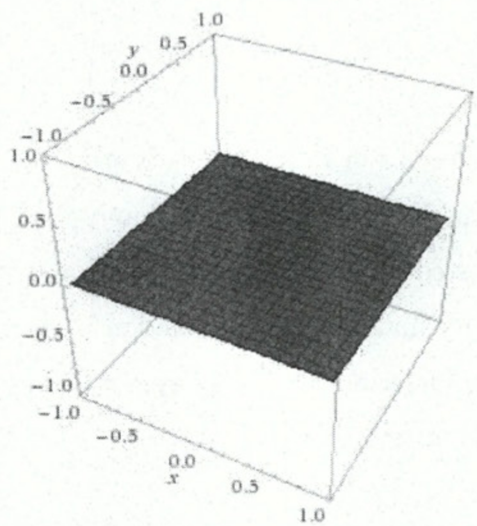

Fig. 5.6. The surface generated by the map $\vec{x}(u, v) = (u, v, 0)$.

We can do the same sort of process for the Minkowski metric, which is just slightly different.

5.1.4 Example: The Minkowski Metric

Recall from Section 3.2 that one of the the conserved quantities under the Lorentz transformation is actually $x_0^2 - x_1^2$ ($x_\mu = (x_0, x_1)$ is the position two-vector). This "length" comes about from a **Minkowski inner product** where, for position two-vectors $x_\mu = (x_0, x_1)$ and $x_\nu = (x_0', x_1')$, we have

$$x_\mu \cdot x_\nu = (x_0, x_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \end{pmatrix} \tag{5.6}$$

and so

$$Q(x_\mu) = x_\mu \cdot x_\mu = (x_0, x_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = x_0^2 - x_1^2.$$

In other words, the Minkowski inner product requires that the metric in Minkoski space G_M to be

$$G_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.7}$$

Obviously, there are no distortions, twists, or stretches in the Euclidean plane, and we discovered that the Euclidean metric is simply the identity matrix. On the other hand, we have suggested that spacetime is curved and we discovered that the metric G_M is not the identity matrix. This distinction is starting to uncover the fact that the metric of a space describes its geometry – whether it is curved or flat. We will investigate this in the next section.

5.2 Curvature

Curvature is a central theme in differential geometry. There are many tools utilized in the field for calculating curvature, but for the purposes of introduction into the topic, we feel it is best to teach via examples in the context of our goal. We intend to be able to detect curvature *as an inhabitant* of a surface S, one who experiences life in the tangent plane $T_p(S)$. An appropriate example seems to be a sphere, since we do live in Earth after all.

5.2.1 Example: Intrinsic Geometry of a Sphere

First, we need to construct a parametrization of the sphere S in order to generate a metric. Let's use the following map of a sphere with radius R

$$\vec{x}(u, v) = (R\cos(u)\cos(v), R\sin(u)\cos(v), R\sin(v)) \tag{5.8}$$

where $-\pi \leq u \leq \pi$ and $-\pi/2 \leq v \leq \pi/2$. The tangent vectors are then

$$\vec{x}_u = (-R\sin(u)\cos(v), R\cos(u)\cos(v), 0),$$

$$\vec{x}_v = (-R\cos(u)\sin(v), -R\sin(u)\sin(v), R\cos(v)).$$

To convince you that this is really a sphere, I have another figure from Mathematica of the parametrized surface below.

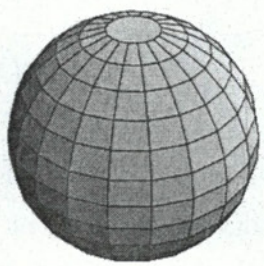

Fig. 5.7. The surface generated by the map in (5.8).

Our friend Bob is an inhabitant of this sphere, so how can he tell the surface is curved without launching a rocket and using an external perspective of the globe? Let's begin with a task. Bob needs to draw a map of a "circular" region of radius d around the North pole. That is, he will start d units away from the North pole and walk in a complete circle to create a region for his map. What would Bob calculate for the circumference C of his region? Clearly, Bob would expect and *calculate* $C = 2\pi d$, the standard equation for the circumference of a circle *in the plane*, before going on with his task. Is that the true circumference of the region on Earth? To *measure* this circumference, we will calculate the length of a curve $\gamma(t) = \vec{x}(u(t), v(t))$ on the surface that follows Bob on his walk. We can parametrize a curve around the North pole as $(u(t), v(t)) = (t, \theta)$ where θ is a constant value for latitude. Recall from Calculus that the length of a curve $\gamma(t)$ is given by

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt, \tag{5.9}$$

where $a \leq t \leq b$. In our case, we need to travel a circuit around the North pole so $-\pi \leq t \leq \pi$ will suffice. Since $\gamma(t) = \vec{x}(u(t), v(t))$, the chain rule gives

$$\gamma'(t) = \frac{d}{dt}\gamma = \partial_u \vec{x}\frac{du}{dt} + \partial_v \vec{x}\frac{dv}{dt} = \frac{du}{dt}\vec{x}_u + \frac{dv}{dt}\vec{x}_v.$$

Notice that we can write the norm in terms of the metric quantities g_{ij}:

$$\begin{aligned} \|\gamma'(t)\| &= \sqrt{\gamma'(t)\cdot\gamma'(t)} \\ &= \sqrt{\frac{du}{dt}\frac{du}{dt}\vec{x}_u\cdot\vec{x}_u + \frac{du}{dt}\frac{dv}{dt}\vec{x}_u\cdot\vec{x}_v + \frac{dv}{dt}\frac{du}{dt}\vec{x}_v\cdot\vec{x}_u + \frac{dv}{dt}\frac{dv}{dt}\vec{x}_v\cdot\vec{x}_v} \\ &= \sqrt{g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2}. \end{aligned} \tag{5.10}$$

According to (5.6), all we need to do is determine the metric G if we want to know the length of $\gamma(t)$. Investigating our tangent vectors $\vec{x}_u$ and $\vec{x}_v$, we find

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} R^2\cos^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}$$

because $v(t) = \theta$ is constant. We finally have the tools to answer the question regarding the measured length around the north pole. Since $(du/dt, dv/dt) = (1, 0)$ due to the parametrization around the North pole, we have

$$g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2 = R^2\cos^2(\theta),$$

and therefore the *measured* circumference of Bob's walk around the North pole is

$$L(\gamma) = \int_{-\pi}^{\pi} \|\gamma'(t)\|\, dt = \int_{-\pi}^{\pi} \sqrt{R^2\cos^2(\theta)}dt = \int_{-\pi}^{\pi} R\cos(\theta)dt = 2\pi R\cos(\theta).$$

The exact value of $\cos(\theta)$ is not the main point here. This result is telling us that we do not live in a flat world, because the *measured* circumference is different than the *calculated* circumference that Bob expected before he started walking. The discrepancy we notice here can be completely described by curvature – very similar to the triangle/hemisphere example given in the beginning of the chapter.

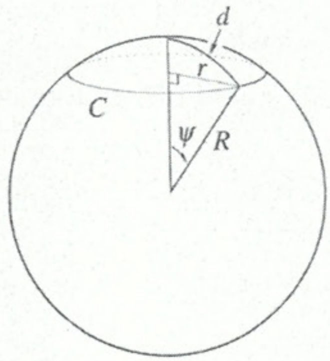

Fig. 5.8. The "circular" map Bob would make is distorted by the geometry of the sphere. Here $\psi = \pi/2 - \theta$ [2].

Of course, Bob could always ask Alice for an extrinsic picture like the one above. Alice could calculate the circumference C with relative ease because, from her perspective, she sees the bent nature of the "circular" region. Alice would determine Bob's *actual* radius from the central axis in Figure 5.8 to be $r = R\cos(\theta)$. Bob's true distance from the North pole d is not the standard Euclidean distance, rather it is $d = R\psi = R(\pi/2 - \theta)$. With the extrinsic perspective, Alice would have known that Bob's *calculated* circumference $2\pi d = 2\pi R(\pi/2 - \theta)$ is greater than his *measured* circumference $2\pi R\cos(\theta)$ (except for $\theta = \pi/2$) long before Bob made his walk around the pole.

A huge accomplishment of intrinsic geometry is the ability to detect curvature without leaving $\mathbb{R}^2$, i.e., without leaving the surface that *locally* seems like a flat plane. But, to be fair, the method of calculation above requires a parametrization map $\vec{x}$ in order to create the metric G. This means there is some extrinsic nature to the process we described, which isn't always achievable. For the moment, forget about the sphere and its parametrization. Instead, let me define the matrix G as follows

$$G = \begin{pmatrix} R^2\cos^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider any path in the uv-plane given by $\gamma(t) = (u(t), v(t))$ where $a \leq t \leq b$. We will now *define* a (new) length of this path to be

$$L_G(\gamma) = \int_a^b \sqrt{g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2}\,dt. \tag{5.11}$$

This length L_G is with respect to the defined G above. We are only considering the uv-plane. If we change the matrix G, we will calculate different lengths from (5.11). Any definition of length that does not conform to the standard Euclidean geometry is then "curved". Note here that if $\gamma(t) = (t, \theta)$ for $-\pi \leq t \leq \pi$, we calculate $L_G = 2\pi R\cos(\theta)$ while the Euclidean distance is still $L_I = 2\pi$ (I denotes the Euclidean metric – the identity matrix).

The second process described is totally *intrinsic* in nature. We do not need a map $\vec{x}$ in order to calculate discrepancies in lengths. All that we were given is a matrix G and defined a length according to (5.11) and could detect "curvature". The method utilizing a parametrization is very helpful when trying to visualize our situation, but sometimes, the completely intrinsic study of geometry is all that is available to us.

6. DESITTER SPACETIME

Our culmination of this material will end with DeSitter spacetime. This is the simplest case of spacetime: the set of all spacelike *unit vectors*. Recall from the **Minkowski norm** that this implies any point $\mathbb{X}$ in 3-dimensional spacetime will satisfy the condition $\|\mathbb{X}\|^2 = -Q(\mathbb{X}) = x^2 + y^2 - (ct)^2 = 1$. This is actually the familiar equation for a hyperboloid of one sheet. We will use the following map to generate such a hyperboloid

$$\mathbb{X}(u, v) = (\sinh(u), \cosh(u)\cos(v), \cosh(u)\sin(v)), \tag{6.1}$$

allowing for $-\infty \leq u \leq \infty$ and $-\pi \leq v \leq \pi$. As you can see in the figure below, the parameter u relates to time and v relates to the position variables x and y. Note that $q^1 = u$ and $q^2 = v$ in this particular case.

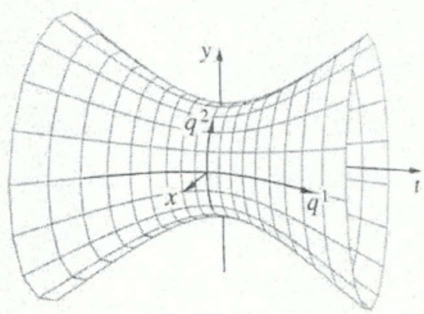

Fig. 6.1. The surface generated by the map $\mathbb{X}$ in (6.1).

Suppose we are to measure the the length of a curve on this surface defined by $\gamma(x) = \vec{x}(u(x), v(x)) = (t_0, x)$ where t_0 is a fixed value of time. In order to do this, we must construct the metric G as we did before. We start by calculating the tangent vectors:

$$\mathbb{X}_u = (\cosh(u), \sinh(u)\cos(v), \sinh(u)\sin(v)), \qquad \text{and}$$

$$\mathbb{X}_v = (0, -\cosh(u)\sin(v), \cosh(u)\cos(v)).$$

Remember the Minkowski inner product we talked about earlier? We are in spacetime coordinates here, so the components of the metric G will need to be generated via the Minkowski inner product rule in (5.6):

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\cosh^2(u) \end{pmatrix}. \tag{6.2}$$

Moving forward with our calculations, we find $\gamma'(x)$ to be

$$\gamma'(x) = \vec{x}_u\left(\frac{du}{dx}\right) + \vec{x}_v\left(\frac{dv}{dx}\right).$$

Recalling that $u(x) = t_0$ and $v(x) = x$ here, this implies $(du/dx, dv/dx) = (0, 1)$, and Minkowski norm is computed using

$$Q(\gamma') = \gamma'(x)\cdot\gamma'(x) = g_{11}\left(\frac{du}{dt}\right)^2 + 2g_{12}\frac{du}{dt}\frac{dv}{dt} + g_{22}\left(\frac{dv}{dt}\right)^2 = -\cosh^2(t_0)$$

and so

$$\|\gamma'\| = \sqrt{-Q(\gamma')} = \cosh(t_0).$$

Therefore, we conclude that $L_G(\gamma)$ grows for $t_0 > 0$ based on the definition in (5.9). We see this as arcs of circles parallel to the xy-plane grow in length (for a constant angle of arc) as time t increases. In this way, one may say space is expanding as time increases. This is a very common result in modern physics – the rapid expansion of the universe.

7. SUMMARY

In this short paper, we have shown what geometry enables us to do: study structures intrinsically. At first intrinsic focus can seem redundant or possibly feel like a hindrance, but the importance of this goal is made clear when dealing with the structure of spacetime. We do not live in spacetime, and we cannot see spacetime. We simply see the physical effects of a strange universe and wonder what could be the cause. In the case of DeSitter spacetime, we were lucky to be able to parametrize the surface. In general, that is not the case.

Let's recap our adventure. First, the introduction of Lorentz transformations transformed classical mechanics quite a bit. We had never heard of speed limits, length contractions, or time dilations before these transformations came to be. Once established, we soon realized that we were in trouble because gravity (or any acceleration) does not mix with flat special relativity. Finally, curved spacetime and general relativity were created in order to fully describe the progression of what we experienced. Although we could not reach the advanced concepts of general relativity, we have laid the groundwork for a future study of the subject.

My hope is that this short summary of results from differential geometry and spacetime have made at least some sense to you. The abilities that geometry grant us are substantial, but the concepts take time set in. Reiterating the purpose of the project, this was merely an introduction into the world of general relativity and geometry. I chose this topic because I found myself very interested in studying these geometrical structures, but finding a resource that was appropriate for my knowledge at the time was very difficult. I hope that, through the use of the examples of analogy, enough interest was sparked for you. If you actually read to this point and you are interested in learning the whole story, it is time to begin studying differential geometry in full. There is so much to learn about tensors, surfaces, mappings, metrics, etc., that simply can't be written in one project introducing spacetime. With differential geometry knowledge, you can really start to understand general relativity. If you enjoyed this, I encourage you to keep going.

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REFERENCES

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